PERIOD-IV

M12CH13

TOPIC

13

Sets and Logic

13.1. INTRODUCTION

In our day-to-day life, we come across collections of objects of a particular type, such as days in a week, months in a year, playing cards in a pack, etc. Such collections are mathematically termed as '*sets*'. Any type of objects can be collected into a set, but set theory is applied more often to objects that are relevant to mathematics. Set theory plays a foundational role in modern mathematics. *Georg Cantor*, a German mathematician, was the founder of set theory, which he defined as:

"*A set is a gathering together into a whole of definite, distinct objects of our perception and of our thought–which are called elements of the set.*"

The elements of a set can be anything: numbers, people, letters of alphabet, other sets, and so on. In this chapter, we shall discuss some basic definitions and operations involving sets.

Sets and Their Representations

In this section, we shall define a set and discuss its representations.

A set is a well-defined collection of objects.

By the phrase '*well-defined collection*', we mean that given a set and an object, it is possible to decide whether or not the object belongs to the set. Also, the decision should not vary from person to person. These objects are called *elements* of the set. Sets are usually denoted by capital letters A, B, C, ... The elements of a set are represented by small letters a, b, c, ...

For example,

(*i*) Consider the collected of first 5 natural numbers. It is a well-defined collection of objected in the sense that we can definitely decide whether a given object belongs to this collection or not. Also, the decision does not vary from to person. We can definitely say that 2 belongs to this collection, but 6 does not.

(*ii*) Consider the collection of 11 best cricket players of the word. It is not a well-defined collection of objects in the sense that we cannot definitely decide whether a given object belongs to this collection or not. Also, the decision may vary from person to person.

Remarks: 1. The term '*Set*' is synonymous with the terms '*Collection*' and '*Class*'.

2. The term '*Element*' is synonymous with the terms '*Object*' and '*Member*'.

Let us now list some of the important and frequently used sets.

$\mathbb{R}$–set of all real numbers

$\mathbb{Z}$–set of all integers

$\mathbb{Q}$–set of all rational numbers

$\mathbb{N}$–set of all natural numbers

$\mathbb{R}^+$–Set of all positive real numbers

$\mathbb{Z}^+$–Set of all positive numbers

$\mathbb{Q}^+$–Set of all positive rational numbers

$\mathbb{C}$–Set of all complex numbers

Notations

We use the following notations and terminology while working with sets:

(*i*) Let a be an element of the set A. We say that "a belongs to A". *The Greek symbol* $\in$ *(epsilon) is used to denote the phrase 'belong to'.*

(*ii*) Let b not be an element of the set A. We say that "*b does not belong to A*" and we write "$b \notin A$".

The symbol $\notin$ is used to denote the phrase "does not belong to".

For example,

(*i*) Let A = {1, 2, 3, 4, 5}. We can say that $2 \in A$, but $6 \notin A$.

(*ii*) Let B = {*a, e, i, o, u*}. We can say that $a \in B$, but $b \notin B$.

Let us not discuss various methods to represent a set.

Representation of a Set

A set be repsented in the following two forms:

1. **Roster Form/Tabular Form:** In this form, we represent a set by listing all the elements of the set, the elements are being separately by commas and enclosed within braces { }.

   For example, let A be the set of all letters in the word 'MATHS'. Then set A can be written in roster form as A = {M, A, T, H, S}.

   Remarks:

   1. The order in which the elements are listed is not important. For example, (M, A, T, H, S} and {S, M, A, T, H} denote the same set.
   2. While writing the set in roster form, an element is not generally repeated, i.e., all the elements are taken as distrinct. For example, the set of all letters in the word 'BOOK' is given by {B, O, K} and not by {B, O, O, K}.

2. **Set-bullder Form.** In this form, we describe the element of the set by using a symbol x which is followed by a colon ":" or vertical bar "|" (read as *such that*). Then we write the characteristic property possessed by the elements of the set and enclose the whole description within braces { }.

   The characteristic property is a wingle common property which is possessed by all the elements of the set but not by any element outside the set.

   For example, if B is the set of all even integers, then set B can be written in set-builder form as

   $$B = \{x : x = 2n,\ n \in \mathbb{Z}\} = \{x \mid x = 2n,\ n \in \mathbb{Z}\}.$$

   In set-builder form, we can use any other symbol, like y, z, etc., instead of x.

   Let us now consider the following examples.

Example 1. *Which of the following are sets? Jusify your answer.*

(*i*) *The collection of all the months of a year beginning with the letter J.*

(*ii*) *All prime numbers less than 11.*

(*iii*) *The collection of all even integers.*

Solution. (*i*) The collection of all the months of a year beginning with the letter J s well-defined because it will always be same for every person. So, it is a set, written as {January, June, July}.

(*ii*) The collection of all prime numbers less than 11 is well-defined because it will always be same for every person. So, it is a set, written as {2, 3, 5, 7}.

(*iii*) The collection of all even integers is well-defined because it will always be same for every person. So, it is a set, written as {..., –4, –2, 0, 2, 4, ...}.

Example 2. *Which of the following are sets? Justify your answers.*

(*i*) *The collection of all rational numbers lying between –2 and 2.*

(*ii*) *The collection of all boys in your class.*

(*iii*) *The collection of question in this chapter.*

Solution.

(*i*) The collection of all rational numbers lying between –2 and 2 is well-defined because it will always be same for every person. So, it is a set.

(*ii*) The collection of all boys in my class is well-defined because it will always be same for every person. So, it is a set.

(*iii*) The collection of questions in this chapter is well-defined because it will always be same for every person. So, it is a set.

Subsets

In this section, we shall discuss an important concept of set theory, viz., subset of a set.

If every element of A is also an element of B, then we say that 'A' is a subset of 'B' or 'A is *content* in B' and we denote it by *writing* $A \subseteq B$.

The symbol '$\subseteq$' is used to denote the phrase '*is subset of*' or '*is contained in*'.

In notational form, we write the following:

$A \subseteq B$, if $a \in A \Rightarrow a \in B$.

The symbol '$\Rightarrow$' is used to denote the word '*implies*', so we read the above statement as "A is a subset of B, If *a* belongs to A implies a belongs to B".

If there exists at least one element of A which is not an element of B, then we say that '*A is not a subste of B*' or '*A is not contained in B*' and we denote it by writing $A \not\subseteq B$.

The symbol '$\not\subseteq$' is used to denote the phrase *'is not a subset of'* or *'is not contained in'*.

Remarks:

1. Every set A is a subset of itself, i.e., $A \subseteq A$.
2. If $A = B$, then $A \subseteq B$ and $B \subseteq A$.
   If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
   Hence, $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.
   We read the above statement as "A and B are equal iff $A \subseteq B$ and $B \subseteq A$".
   The symbol '$\Leftrightarrow$' is used for two-way implication, and is usually read as *'if and only if'* (briefly written as *'iff'*).
3. The empty set is a subset of every set, i.e., $\phi \subseteq A$ for every set A. This is because there is no element in ϕ which does not belong to A.
4. If A is a subset of B, then we can also say that B is *superset* of A.
5. If $A \subseteq B$ and $A \neq B$, then A is called a *proper subset* of B and we denote it by $A \subset B$.
6. If A is not a proper subset of B, then we denote it by writing $A \not\subset B$.

Subset of $\mathbb{R}$

Some of the important subsets of the set $\mathbb{R}$ of the real numbers are:

(*i*) The set of all natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$.

(*ii*) The set of all integers, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

(*iii*) The set of all rational numbers,

$$\mathbb{Q} = \left\{x : x = \frac{p}{q};\ p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$$

(*iv*) The set of all irrational numbers, $\mathbb{T} = \{x : x \in \mathbb{R} \text{ and } x \notin \mathbb{Q}\}$.

Clearly, we have $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}$, $\mathbb{T} \subseteq \mathbb{R}$, $\mathbb{N} \not\subseteq \mathbb{T}$ and $Z \not\subseteq \mathbb{T}$.

Intervals as Subsets of $\mathbb{R}$

Let us now discuss the various types of intervals which are subsets of the set $\mathbb{R}$ of all real number. Then are four types of intervals:

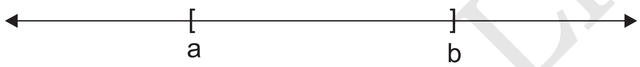
(*i*) **Open Interval.** Let $a, b \in \mathbb{R}$ such that $a < b$. Then, the set of all real numbers between a and b excluding both the a and b is called an *open interval from a to b.* It is denoted by (a, b).

It can be shown with the help of the following diagram:

(*ii*) **Closed Interval.** Let $a, b \in \mathbb{R}$ such that $a < b$. Then, the set of all real numbers between a and b including both the a and b is called a *closed interval from a to b*. It is denoted by (a, b).

$$[a, b] = \{x : x \in \mathbb{R} \text{ and } a \leq x \leq b\}.$$

It can be shown with the help of the following diagram:



(*iii*) **Left Closed-Right Open Interval.** Let $a, b \in \mathbb{R}$ such that $a < b$. Then, the set of all real numbers between a and b including both a but excluding b, is called a *left-closed-right open interval from a to b*. It is denoted by $[a, b)$.

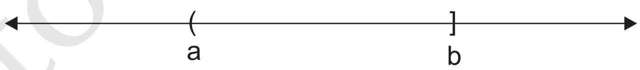
$$[a, b) = \{x : x \in \mathbb{R} \text{ and } a \leq x \leq b\}.$$

It can be shown with the help of the following diagram:

(*iv*) **Left Open Right-Closed Interval.** Let $a, b \in \mathbb{R}$ such that $a < b$. Then, the set of all real numbers between a and b excluding a but including b, is called a *left-open-right closed interval from a to b*. It is denoted by $(a, b]$.

$$(a, b] = \{x : x \in \mathbb{R} \text{ and } a \leq x \leq b\}.$$

It can be shown with the help of the following diagram:



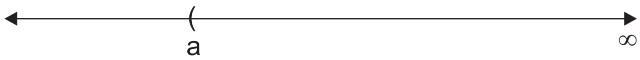
The length of each of the interval $(a, b]$, $[a, b)$ and $(a, b]$ is $b - a$.

There are a few more types of intervals. There are as follows:

(*i*) **Open Right-Ray.** Let $a \in \mathbb{R}$. Then, the set of all real numbers greater than a is called an *open right-ray from a*. It is denoted by (a, ∞).

$$[a, \infty) = \{x : x \in \mathbb{R} \text{ and } a < x < \infty\}.$$

It can be shown with the help of the following diagram:



(*ii*) **Closed Right-Ray.** Let $a \in \mathbb{R}$. Then, the set of all real numbers greater than or equal to a is called a *closed right-ray from a.* It is denoted by (a, ∞).

$$[a, \infty) = \{x : x \leq \mathbb{R} \text{ and } a \leq x < \infty\}.$$

It can be shown with the help of the following diagram:

(*iii*) **Open Left-Ray.** Let $a \in \mathbb{R}$. Then, the set of all real numbers less than a is called an *open left-ray from a.* It is denoted by $(-\infty, a)$.

$$(-\infty, a,) = \{x : x \in \mathbb{R} \text{ and } -\infty < x < a\}.$$

It can be shown with the help of the following diagram:

(*iv*) **Closed Left-Ray.** Let $a \in \mathbb{R}$. Then, the set of all real numbers less than or equal to a is called an *closed left-ray from a.* It is denoted by $(-\infty, a)$.

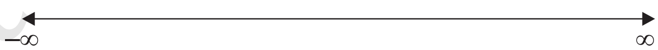
$$(-\infty, a] = \{x : x \in \mathbb{R} \text{ and } -\infty < x < a\}.$$

It can be shown with the help of the following diagram:

(*v*) **The Real Line.** Then, set of all real numbers is called *the real line.* It is denoted by $(-\infty, \infty)$.

$$(-\infty, \infty,) = \{x : x \in \mathbb{R}\}.$$

It can be shown with the help of the following diagram:



Power Set

Let A be any set. Then, we can talk about the subsets of A. Let us collect all the subsets of A to form a new set. This new set is known as *power set* of the set A. We have the following definition:

The collection of all subsets of set A is called the *power set of A*. It is denoted by P(A).

For example, consider the set A = $\{a, b\}$.

Then, $\phi\{a\}$, $\{b\}$ and $\{a, b\}$ are subsets of A.

So, the power set of A is given by P(A) = $\{\phi, \{a\}, \{b\}, \{a, b\}\}$.

Remarks:

1. In a power set, every elements is a set.
2. If A = ϕ, then P(A) has just one element, viz., ϕ.
3. Let A = $\{a, b\}$. Then, the set of all proper subsets of A is $\{\phi, \{a\}, \{b\}\}$.
4. If a set A has n elements, then the number of elements in the power set P(A) is 2^n.
5. If a set A has n elements, then the number of proper subsets of A is one less than the number of elements in the power set P′(A), i.e., $2^n - 1$.

Universal Set

In any discussion in set theory, there is always a set that contains all the sets under consideration, i.e., there is always a set that is a superset of each of the sets under consideration. Such a set is called *universal set.* We have the following definition:

A set contains all sets under consideration is called *universal set.* It is denoted by U.

For example, If A = {1, 2, 3} and B = {5, 6, 8} are the sets under consideration, then the set given by U = {1, 2, 3, 4, 5, 6, 7, 8} can be taken as universal set. The choice of universal set is not unique. The set U = {1, 2, 3, 5, 6, 8} can also be taken as universal set.

Let us now understand the above concepts with the help of the following examples.

Example 3. *Let A = {a, e, i, o, u} and B = {a, b, c, d}. Is A subset of B? Is B a subset of A?*

Solution. Given that A = $\{a, e, i, o, u\}$ and B = $\{a, b, c, d\}$.

Since, $e \in$ A and $e \notin$ B. So, A is not a subset of B.

Since, $b \in$ B and $b \notin$ A. So, B is not a subset of A.

Types of Sets

In this section, we shall discuss some important types of sets.

1. **Empty Set/Null Set/Void Set.** A set that does not contain any element is called the *empty set* or the *null set* or the *void set.* The empty set is denoted by the symbol ϕ (phi)or { }.
2. **Non-empty Set.** A set that contains at least one element is called a *non-empty set.* If a set A is non-empty, then we denote it by writing $A \neq \phi$.
3. **Singleton Set.** A set that contain exactly one element is called *singleton set.*

For example,

(*i*) Consider the set, $A = \{x \in \mathbb{N} : x < 0\}$. Since, there is natural number x which is negative. So, A does not contain any element. Hence, A is the empty set, i.e., $A = \phi$.

(*ii*) Consider the set, $B = \{x \in \mathbb{N} : 2 < x < 5\}$. Since, $3 \in B$, is non-empty set, *i.e.*, $B \neq \phi$.

(*iii*) Consider the set, $C = \{x \in \mathbb{N} : 2 < x < 4\}$. Then $C = \{3\}$, i.e., C is constants exactly one element. Hence, C is a singleton set.

4. **Finite Set.** A set which is empty or consists of a definite number of elements is called a *finite set.*
5. **Infinite Set.** A set which does not consists of a definite number of elements is called a *infinite set.* All infinite sets cannot be described in the roster form.

For example,

(*i*) Consider the set, $A = \{x \in \mathbb{N} : 2 < x < 3\}$. Since there is no natural number x which lies between 2 and 3. So set A does not contain any element, i.e., $A = \phi$. Hence, A is a finite set.

(*ii*) Consider the set, $B = \{x \in \mathbb{N} : 2 < x < 10\} = \{3, 4, 5, 6, 7, 8, 9\}$. Since set B contains definite number of elements (7 elements, to be precise). So, set B is a finite set.

(*iii*) Consider the set, C = *Set of all rivers in India.* Since, set C contains definite number of elements. So, set C is a finite set.

(*iv*) Consider the set, D = *Set of all natural numbers.* Since, set D does not contain definite number of elements. It can be expressed in roster form as $D = \{1, 2, 3, ...\}$.

(*v*) Consider the set, E = *Set of all real numbers.* Since, set E does not contain definite number of elements. So, set E is a infinite set. It cannot be expressed in roster form.

Remarks:

1. Empty set and singleton set are finite sets.
2. The set-builder form form of empty set is $\{x : x \neq x\}$.
3. Some people say that set $\mathbb{N}$ of all natural numbers has a definite number of elements, viz., *infinity*. This is incorrect, as *'infinity is not a number'*.

6. Cardinal Number (or order) of a Finite Set. The number of distinct elements of a finite set is called *cardinal number* or *order* of that finite set. The cardinal number of a set A is denoted by $n(A)$. The cardinal number of the empty set is 0. The cardinal number of a singleton set is 1. The cardinal number of an infinite set is *not defined.*

For example,

(*i*) Consider the set, $A = \{x \in \mathbb{N} : 2 < x < 3\} = \{\}$. Since, A is empty set. So, $n(A) = 0$.

(*ii*) Consider the set, $B = \{x \in \mathbb{N} : 2 < x < 10\} = \{3, 4, 5, 6, 7, 8, 9\}$. Since, B contain 7 elements. So, $n(B) = 7$.

(*iii*) Consider the set, C = *set of all natural numbers* = $\{1, 2, 3, ...\}$. Since, C is an infinite set. So, $n(C)$ is not defined.

Remark: The set of all persons in the world is a finite set, but the number of elements in the set is so large that it is very difficult to find the number. So, the cardinal number of a set is sometimes very difficult to find, but it does not mean that the cardinal number is not defined for such sets.

7. Equal Sets. Two sets A and B are said to be *equal sets* if they have exactly the same elements and we write A = B.

8. Unequal Sets. Two sets A and B are said to be *unequal sets* if they are not equal and we write $A \neq B$.

For example,

(*i*) Consider the pair of sets, $A = \{1, 2, 3, 4\}$ and $B = \{4, 3, 2, 1\}$. Since A and B have exactly the same elements. So, A = B.

(*ii*) Consider the pair of sets, $C = \{1, 2, 3, 4, 5\}$ and $D = \{4, 3, 2, 1\}$. Since C contains 5, but D does not. So, $C \neq D$.

(*iii*) Consider the pair of sets, $E = \{1, 2, 3, 4\}$ and $F = \{1, 2, 3, 5\}$. Since E contains 4, but F does not. So, $E \neq F$.

9. Equivalent Sets. Two finite sets A and B are said to be *equivalent sets* if they have the same number of elements, i.e., if $n(A) = n(B)$. We denote it by writing $A \leftrightarrow B$.

For example,

(*i*) Consider the pair of sets, A = {1, 2, 3, 4} and B = {3, 2, 4, 1}. Since, $n(A) = n(B)$. So, $A \leftrightarrow B$. We also observe that A = B.

(*ii*) Consider the pair of sets, C = {*a*, *b*, *c*} and D = {1, 2, 3}. Since, $n(C) = n(D)$. So, $C \leftrightarrow D$. We also observe that $C \neq D$.

(*iii*) Consider the pair of sets, E = {1, 2, 3, 4} and F = {*a*, *b*, *c*}. Since, $n(E) \neq n(F)$. So, E and F are not equivalent sets.

Remark: Equal sets are always equivalent but equivalent sets may or may not be equal.

Let us now understand the above types of sets with the help of following examples.

Example 4. *State which of the following sets are empty or non-empty sets:*

(*i*) *Set of all even prime numbers.*

(*ii*) *Set of all even prime numbers > 2.*

(*iii*) *Set of all circle passing through (0, 0).*

(*iv*) $\{x : 1 < x < 2, x \text{ is a natural number}\}$

(*v*) $\{x : x^2 - 2 = 0 \text{ and } x \text{ is rational number}\}$.

(*vi*) *{x : x is a point common to any two parallel lines}.*

(*vii*) $\{x : x^2 = 4, x \text{ is odd}\}$.

(*viii*) $\{x : x^2 = 4, x \text{ is odd}\}$.

Solution.

(*i*) Non-empty set (*ii*) Empty set (*iii*) Non-empty set

(*iv*) Empty set (*v*) Empty set (*vi*) Empty set

(*vii*) Empty set (*viii*) Non-empty set.

13.2. VENN DIAGRAM V/S EULER DIAGRAM

The relationships between the sets can be easily represented by means of diagrams. There are two types of representations, viz, *Venn Diagrams and Eular Diagrams.* Although both the diagrams look very similar, there are some subtle differences between them. The terms Venn diagram and Euler diagram are often confused. This section will clear the doubts about Venn diagram v/s Euler diagrams.

Similarity

The similarity between the two diagrams is that both of them consists of rectangules and closed curves usually circles). The universal set is represented by a rectangule and its subsets by circle. The interior of the circle symbollically represents the elements of the set, while the exterior represents elements that are not numbers of the set.

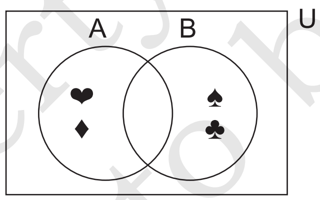
Differences

(*i*) A Venn diagram shows all possible logical relations between a finite collection of sets, whereas an Euler diagarm does not necessarily show all possible intersections of the sets.

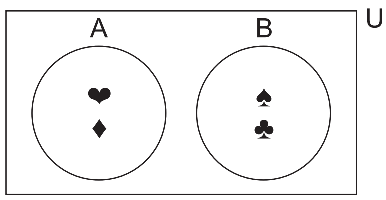
(*ii*) In a Venn diagram the circles always intersect, whereas in an Eular diagram the circles may or may not intersect.

(*iii*) In a Venn diagram two disjoint sets are represented with overlapping circles and the common region represents the empty set, whereas in an Eular diagram the disjoint sets are represented with non-overlapping circles.

Let us consider a very simple example. Let U denote the universal set of 52 playing cards. Let A and B denote the set of all red cards and the set of all black cards respectively.

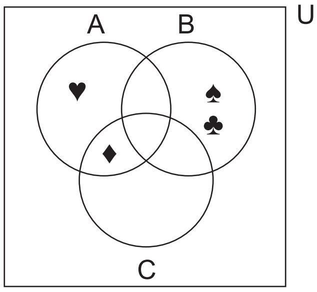


Venn diagram

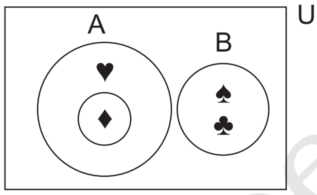


Euler diagram

Let us consider another simple example. Let U denote the universal set of 52 playing cards. Let A, B and C denote the set of all red cards, the set of all black cards and the set of all diamond cards respectively.



Venn diagram



Euler diagram

A Venn diagram shows an intersection between the two sets even though that intersection can be any empty set. Euler diagram, on the other hand, doesn't show an intersection.

However, we will restrict our presentation of subject matter to the application of Venn dagram only, as required in terms of the syllabus.

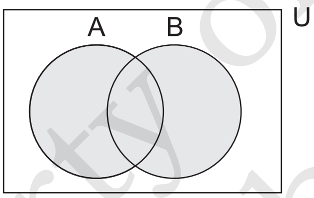
Let us now consider the following examples.

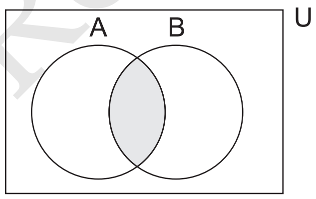
Examples 5. *Draw appropriate Venn diagram for each of the following:*

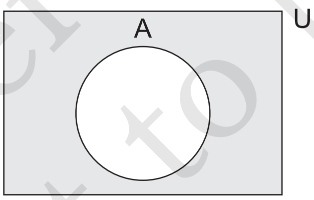
(*i*) $A \cup B$ (*ii*) $A \cap B$ (*iii*) A' (*iv*) $A - B$

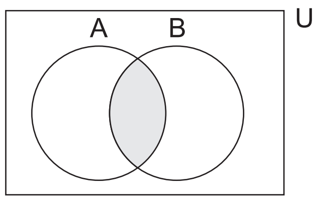
(*v*) $A \Delta B$.

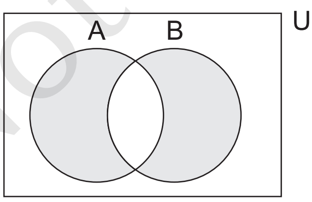
Solution. Let U be the universal set. The required region is shown shaded in the graph.

(*i*)


(*ii*)


(*iii*)


(*iv*)


(*v*)


13.3. OPERATION ON SETS

We are already familier with the operations of addition, subtraction, multiplication and division of numbers. In the following sections, we shall discuss some basic operations on sets, viz., union of two sets intersection of two sets, complement of a set, difference of two sets and symmetric difference of two sets. Hence forth, we will refer all our sets as subsets of some universal set.

Union of Sets

In this section, we shall discuss the union operation on sets.

Let A and B be any two sets. Then, the *union* of A and B is the set which consists of all above elements which are earlier in A or in B or in both.

The symbol '$\cup$' is used to denote the *union* of sets.

Symbolically, we denote the union of A and B with $A \cup B$ (read as 'A *union* B') and we can write $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Some important properties of the union operation are:

1. For any set A, we have $A \cup U = U$.
2. For any set A, we have $A \cup \phi = A$.
3. (Idempotent Law) For any set A, we have $A \cup A = A$.
4. (Commutative Law) For any two sets A and B, we have $A \cup B = B \cup A$.
5. (Associative Law) For any three sets A, B and C. We have
$$A \cup (B \cup C) = (A \cup B) \cup C$$
6. For any two sets A and B such that $A \subseteq B$, we have $A \cup B = B$.

Proof:

1. $A \cup U = \{x : x \in A \text{ or } x \in U\} = \{x : x \in U\} = U$.
2. $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\} = A$.
3. $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$.
4. $A \cup B = \{x : x \in A \text{ or } x \in B\} = \{x : x \in B \text{ or } x \in A\} = B \cup A$.
5. $A \cup (B \cup G) = \{x : x \in A \text{ or } x \in B \cup C\} = \{x : x \in A \text{ or } (x \in B \text{ or } x \in C)\}$
$$= \{x : (x \in A \text{ or } x \in B) \text{ or } x \in C\}$$
$$= \{x : x \in A \cup B \text{ or } x \in C\}$$
$$= (A \cup B) \cup C$$

6. $A \cup B = \{x : x \in A \text{ or } x \in B\}$
$= \{x : x \in B\} = B$

Remarks:

1.If $x \in A \cup B$, then $x \in A$ or $x \in B$.

2.If $x \notin A \cup B$, then $x \notin A$ and $x \notin B$.

3.$A \subseteq A \cup B$ and $B \subseteq A \cup B$.

4.For there sets A, B and C, we have

$$A \cup B \cup C = \{x : x \in A \text{ or } x \in B \text{ or } x \in C\}.$$

Let us now consider the following examples.

Example 6. *Let A = {a, e, i, o, u} and B = {a, b, c}. Find A ∪ B.*

Solution. Given A = {*a*, *e*, *i*, *o*, *u*} B = {*a*, *b*, *c*}

Then, $A \cup B = \{a, e, i, o, u\} \cup \{a, b, c\} = \{a, i, e, o, u, b, c\}$.

Intersection of Sets

In this section, we shall discuss the intersection operation on sets.

Let A and B be any two sets. Then, the *intersection* of A and B is the set which consits of all those elements which are in both A and B.

The symbol '∩' is used to denote the *intersection* of sets.

The denote the intersection of A and B with A ∩ B (read as '*A intersection B*') and we can write

$$A \cap B = \{x : x \in A \text{ and } x \in B).$$

Some important properties of the intersections operation are:

1. For any set A, we have $A \cap U = A$

2. For any set A, we have $A \cap \phi = \phi$.

3. (Idempotent Law) For any set A, we have $A \cap U = A$

4. (Commutative Law) For any two sets A and B, we have $A \cap B = B \cap A$.

5. (Associative Law) For any three sets A, B and C, we have $A \cap (B \cap C) = (A \cap B) \cap C$.

6. (Distributive Law) For any three sets A, B and C, we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

7. (Distributive Law) For any three sets A, B and C, we have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

8. For any two sets A and B such that $A \subseteq B$, we have $A \cap B = A$.

Proof:

1. $A \cap U = \{x : x \in A \text{ and } x \in U\} = \{x : x \in A\} = A$.

2. $A \cap o = \{x : x \in A \text{ and } x \in \phi\} = \phi$

3. $A \cap A = \{x : x \in A \text{ and } x \in A\} = \{x : x \in A\} = A$.

4. $A \cap B = \{x : x \in A \text{ and } x \in B\} = \{x : x \in B \text{ and } x \in A\} = B \cap A$.

5. $A \cap (B \cap U) = \{x : x \in A \text{ and } B \cap C\}$

$= \{x : x \in A \text{ and } (x \in B \text{ and } x \in C)\}$

$= \{x : (x \in A \text{ and } x \in B) \text{ and } x \in C\}$

$= \{x : x \in A \cap B \text{ and } x \in C\} = (A \cap B) \cap C$.

6. $A \cup (B \cap C) = \{x : x \in A \text{ and } B \cap C\}$

$= \{x : x \in A \text{ and } (x \in B \text{ and } x \in C)\}$

$= \{x : (x \in A \text{ and } x \in B) \text{ and } (x \in A \text{ or } x \in C)\}$

$= \{x : x \in A \cap B \text{ and } x \in C\} = (A \cap B) \cap C$.

7. $A \cap (B \cup C) = \{x : x \in A \text{ and } B \cup C\}$

$= \{x : x \in A \text{ and } (x \in B \text{ and } x \in C)\}$

$= \{x : (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)\}$

$= \{x : x \in A \cap B \text{ and } x \in A \cap C\} = (A \cap B) \cup (A \cap C)$.

8. $A \cap B = \{x : x \in A \text{ and } x \in B\}$

$= \{x : x \in A\}$ $\quad [\because \quad A \subseteq B]$

$= A$.

Remarks:

1. If $x \in A \cap B$, then $x \in A$ and $x \in B$.
2. If $x \notin A \cap B$, then $x \notin A$ or $x \notin B$.
3. $A \cap B \subseteq A$ and $A \cap B \subseteq B$.
4. For three sets A, B and C, we have $A \cap B \cap C = (x : x \in A \text{ and } x \in B \text{ and } x \in C\}$.
5. Two sets A and B are said to be *disjoint* if $A \cap B = \phi$.

Let us now consider the following examples:

Example 7. *Let A = {1, 2, 3} and B =* ϕ. *Find* $A \cap B$.

Solution. Given $A = \{1, 2, 3\}$ and $B = \phi = \{\}$.

Then, $A \cap B = \{1, 2, 3\} \cap \{\} = \{\}$.

Complement of a Set

In this section, we shall discuss the complement operation on sets.

Let U be the universal set and let A be any set. Then, the *complement of A* is the set which consists of all those elements which are in U but not in A.

Symbolically, we denote the complement of A with A′ or A^c (read as '*A complement*') and we can write

$$A' = \{x : x \in U \text{ and } x \notin A\}.$$

Some important properties of the complement operation are:

1. $\phi' = U$.
2. $U' = \phi$.
3. (Complement Laws) For any set A, we have $A \cup A' = U$ and $A \cap A' = \phi$.
4. (Double Complementation Law) For any set A, we have $(A')' = A$.
5. (De Morgan's Law) For any two sets A and B, we have

$$(A \cup B)' = A' \cap B' \text{ and } (A \cap B)' = A' \cup B'.$$

Proof:

1. $\phi' = \{x : x \in U \text{ and } x \notin \phi\} = \{x : x \in U\} = U$.
2. $U' = \{x : x \in U \text{ and } x \notin U\} = \phi$.
3. $A \cup A' = \{x : x \in A \text{ or } x \in A'\} = \{x : x \in A \text{ or } x \notin A\} = U$.
   $A \cap A' = \{x : x \in A \text{ and } x \in A'\} = \{x : x \in A \text{ or } x \notin A\} = \phi$.
4. $(A')' = \{x : x \in U \text{ and } x \notin A'\} = \{x : x \in U \text{ and } x \in A\} = A$.
5. $(A \cup B)' = \{x : x \in U \text{ and } x \notin A \cup B\}$
   $= \{x : x \in U \text{ and } (x \notin A \text{ and } x \notin B)\}$
   $= \{x : (x \in U \text{ and } x \notin A) \text{ and } (x \in U \text{ and } x \notin B)\}$
   $= \{x : x \in A' \text{ and } x \in B'\}$
   $= A' \cap B'$.

   $(A \cap B)' = \{x : x \in U \text{ and } x \notin A \cap B\}$
   $= \{x : x \in U \text{ and } (x \notin A \text{ or } x \notin B)\}$
   $= \{x : (x \in U \text{ and } x \notin A) \text{ or } (x \in U \text{ and } x \notin B)\}$
   $= \{x : x \in A' \text{ or } x \in B'\} = A' \cup B'$.

Note: The De Morgan's laws have been named after a British mathematician *Augustus De Morgan*, who has formulated these laws. These laws can be expressed as follows:

(*i*) The complement of the union of two sets is the intersection of their complements.

(*ii*) The complement of the intersection of two sets is the union of their complements.

Let us now consider the following examples.

Example 8. *Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and A = {1, 3, 5, 7, 9}. Find A′.*

Solution. Given U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and A = {1, 3, 5, 7, 9}. Then A′ = {2, 4, 6, 8, 10}.

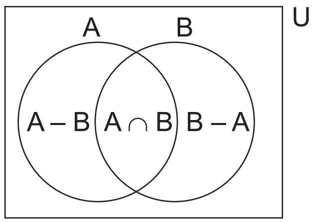
Two-Set Problems

These section. We shall discuss the problems in which two sets are under consideration. Let us now state and gave a very important theorem.

Theorem: If A and B are any two finite sets, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Proof: Give that A and B are two finite sets. From the Venn diagram, it is clear that A – B, A ∩ B and B – A are three disjoint sets.



When the union of these three sets is equal to A ∪ B.

$$\begin{aligned} n(A \cup B) &= n(A - B) + n(A \cap B) + n(B - A) \\ &= n(A - B) + n(A \cap B) + n(B - A) + n(A \cap B) - n(A \cap B) \\ &= [n(A - B) + n(A \cap B)] + [n(B - A) + n(A \cap B)] - n(A \cap B) \\ &= n(A) + n(B) - n(A \cap B). \end{aligned}$$

this prove the theorem.

Corollary. If A and B are finite and disjoint, then

$$n(A \cup B) = n(A) + n(B).$$

In this following table, we give some verbel descriptions with their equivalent set theoretical notation (involving two sets A and B).

Verbal Description	*Set Theoretical Notation*
not A	A'
only A	$A \cap B'$
A but not B	$A \cap B'$
only B	$A' \cap B'$
B but not A	$A' \cap B$
either A or B	$A \cup B$
At least one of A and B	$A \cup B$
A and B	$A \cap B$
neither A and B	$A' \cap B'$

Let us now consider the following examples.

Example 9. *If X and Y are two sets such that X has 21 elements, Y has 32 elements, and $X \cap Y$ has 11 elements, how many elements does $X \cup Y$ have?*

Solution. We have, $n(X) = 21$, $n(Y) = 32$, $n(X \cap Y) = 11$, $n(X \cup Y) = ?$

Now, $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y) = 21 + 32 - 11 = 42$

Hence, number of elements in $X \cup Y$ is 42.

Three-Set Problems

In this section, we shall discuss the problems in which three sets are under consideration. Let us now state and prove a very important theorem.

Theorem. If A, B and C are any three finite sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C).$$

Proof. Given that A, B and C are three finite sets. Then,

$$\begin{aligned} n(A \cup B \cup C) &= n[A \cup (B \cup C)] \\ &= n(A) + n(B \cup C) - n[A \cap (B \cup C)] \\ &= n(A) + [n(B) + n(C) - n(B \cap C)] - n[A \cap (B \cup C)] \\ &= n(A) + n(B) + n(C) - n(B \cap C) - n[(A \cap B) \cup (B \cap C)] \\ &= n(A) + n(B) + n(C) - n(B \cap C) \\ &\quad - n[(A \cap B) + n \cup (A \cap C) - n(A \cap B \cap A \cap C)] \\ &= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) \\ &\quad - n(A \cap C) + n(A \cap B \cap C). \end{aligned}$$

In the following table, we give some verbal description with their equivalent set theoretical notation (involving three sets A, B and C).

Verbal Description	Set Theoretical Notation
not A	A'
only A	$A \cap B' \cap C'$
only B	$A' \cap B \cap C'$
only C	$A' \cap B' \cap C$
only A and B	$A \cap B \cap C'$
only B and C	$A' \cap B \cap C$
only A and C	$A \cap B' \cap C$
at least one of A, B and C	$A \cup B \cup C$
none of A, B and C	$A' \cap B' \cap C'$

Example 10. *In a survey of 25 students, it was found that 15 had taken Mathematics, 12 had taken Physics and 11 had taken Chemistry, 5 had taken Mathematics and Chemistry, 9 had taken Mathematics and Physics, 4 had taken Physics and Chemistry and 3 had taken all the three subjects. Find the number of students who had taken*

(*i*) *only Chemistry*
(*ii*) *only Mathematics.*
(*iii*) *only Physics*
(*iv*) *only one of the subjects.*
(*v*) *only Physics and Chemistry*
(*vi*) *only two of the subjects.*
(*vii*) *only Mathematics and Chemistry.*
(*viii*) *only two of the subjects.*
(*ix*) *at least one subject.*
(*x*) *none of the subject.*

Solution. Let U denote the set of all students,

A denote the set of students who had taken Mathematics,

B denote the set of students who had taken Physics and

C denote the set of students who had taken Chemistry.

We have, $n(U) = 25$, $n(A) = 15$, $n(B) = 12$, $n(C) = 11$, $n(A \cap C) = 5$,
$n(A \cap B) = 9$, $n(B \cap C) = 4$, $n(A \cap B \cap C) = 3$.

(*i*) $n(C \cap B' \cap A') = ?$

Now, $n(C \cap B' \cap A') = n(C) - n(C \cap B) - n(C \cap A) + n(C \cap B \cap A)$

$= 11 - 4 - 5 + 3 = 5.$

Hence, number of students who had taken only Chemistry is 5.

(*ii*) $n(A \cap C' \cap B') = ?$

Now, $n(A \cap C' \cap B') = n(A) - n(A - C) - n(A \cap B) + n(A \cap C \cap B)$

$= 15 - 5 - 9 + 3 = 4.$

Hence, number of students who had taken only Chemistry is 4.

(*iii*) $n(B \cap C' \cap A') = ?$

Now, $n(B \cap C' \cap A') = n(B) - n(B - C) - n(B \cap A) + n(B \cap C \cap A)$

$= 12 - 4 - 9 + 3 = 4.$

Hence, number of students who had taken only Chemistry is 2.

Now $n(C \cap B' \cap A') + n(A \cap C' \cap B') + n(B \cap C' \cap A') = ?$

Now, $n(C \cap B' \cap A') + n(A \cap C' \cap B') + n(B \cap C' \cap A')$

$= n(A) + n(B) + n(C) - 2[n(A \cap B) + n(A \cap C) + n(B \cap C)$

$+ 3[n(A \cap B \cap C)]$

$= 15 + 12 + 11 - 2(9 + 5 + 4) + 3(3) = 11.$

Hence, number of students who had taken only one subject is 11.

Example 11. *Write the following sets in the roster form:*

(*i*) $A = \{x : x \in \mathbb{R}, 2x + 11 = 15\}$ (*ii*) $B = \{x : x^2 = x \in \mathbb{R}\}$

(*iii*) $C = \{x : x$ *is a positive factor of a prime number* $p\}$

Solution. (*i*) $\{2\}$ (*ii*) $\{0, 1,\}$ (*iii*) $\{1, p\}$.

Example 12. *Let* $T = \left\{x : \frac{x+5}{x-7} - 5 = \frac{4x-40}{13-x}\right\}$. *Is T an empty set? Justify your answer.*

Solution. Given $T = \left\{x : \frac{x+5}{x-7} - 5 = \frac{4x-40}{13-x}\right\}$.

Now, $\frac{x+5}{x-7} - 5 = \frac{4x-40}{13-x}$

$\Rightarrow \frac{x+5}{x-7} = \frac{4x-40}{13-x}$

$\Rightarrow (40 - 4x)(13 - x) = (4x - 40)(x - 7)$

$\Rightarrow \quad (40 - 4x)(13 - x) - (4x - 40)(x - 7) = 0$

$\Rightarrow \quad 6(40 - 4x) = 0$

$\Rightarrow \quad x = 10$

Thus, $\quad T = \left\{x : \frac{x+5}{x-7} - 5 = \frac{4x - 40}{13 - x}\right\} = \{10\}$

Hence, T is a non-empty set.

EXERCISE

1. Write the following in roster form:

(*i*) $\{a_n : n \in \mathbb{N}, a_{n+1} = 3a_n \text{ and } a_1 = 1\}$.

(*ii*) $\{a_n : n \in \mathbb{N}, a_{n+2} = a_{n+1} + a_n \text{ and } a_1 = a_2 = 1\}$.

2. Decide, among the following sets, which sets are subsets of one and another:

$A = \{x : x \in \mathbb{R}$ and x satisfies $x^2 - 8x + 12 = 0\}$, $B = \{2, 4, 6\}$, $C = \{2, 4, 6, 8, \ldots\}$, $D = \{6, 6\}$.

3. If A = {3, 5, 7, 9, 11}, B = {7, 9, 11, 13}, C = {11, 13, 15} and D = {15, 17}; find

(*i*) $A \cap B$ (*ii*) $A \cap C \cap D$. (*iii*) $A \cap (B \cup D)$

(*iv*) $(A \cap B) \cap (B \cup C)$ (*v*) $(A \cup D) \cap (B \cup C)$.

4. If Y = {1, 2, 3, ..., 10}, and a represents and element of Y, write the following sets, containing all the elements satisfying the given conditions.

(*i*) a is less than 6 and $a \in Y$ (*ii*) $a + 1 = 6, a \in Y$

(*iii*) $a \in Y$ but $a^2 \notin Y$.

5. Write the following sets in roster form:

(*i*) $\{x : x$ is an integer and $-3 < x \leq 5\}$.

(*ii*) $\{x : x$ is a two-digit natural number having sum of digits as $8\}$.

(*iii*) $\{x : x$ is a prime number which is divisor of $60\}$.

(*iv*) $\left\{x : x = \frac{2n+1}{n+1}, n \in \mathbb{N} \text{ and } n < 5\right\}$.

6. Write the following sets in roster form:

(*i*) The set of all even integer lying betwee – 5 and 5.

(*ii*) The set of all letters in the word 'TRIGONOMETRY'.

(*iii*) The set of all digits in our decimal system.

(*iv*) The fractions whose numerator is 3 and denominator is an even natural number less than 10.